Gradient Flows and Geometric Active Contour Models

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Abstract

In this paper, we analyze the geometric active contour models discussed in [6, 18] from a curve evolution point of view and propose some modifications based on gradient flows relative to certain new feature-based Riemannian metrics. This leads to a novel snake paradigm in which the feature of interest may be considered to lie at the bottom of a potential well. Thus the snake is attracted very naturally and efficiently to the desired feature. Moreover, we consider some 3-D active surface models based on these ideas.

1 Introduction

In the past few years, a number of approaches have been proposed for the problem of *snakes* or *active contours*. The underlying principle in these works is based upon the utilization of deformable contours which conform to various object shapes and motions. Snakes have been used for edge and curve detection, segmentation, shape modelling, and visual tracking. The recent book by Blake and Yuille [5] contains an excellent collection of papers on the theory and practice of deformable contours together with a large list of references to which which we refer the interested reader.

In most of the classical frameworks, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain contraints set by the user. See [14, 29, 5]. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters.

In the present paper, we consider a method which was strongly influenced by the elegant approaches of Caselles *et al.* [6] and Malladi *et al.* [18]. In these works, a level set curve evolution method is presented to solve the problem. Our idea is simply to note that both these approaches are based on Euclidean curve shortening evolution which in turn defines the gradient direction in which the Euclidean perimeter is shrinking as fast as possible. (See Section 2.) Pushing this concept to the next logical step, we can derive new active contour models by multiplying the Euclidean arc-length by a function tailored to the features of interest to which we want to flow, and then writing down the resulting *gradient evolution equations.* Mathematically, this amounts to defining a new Riemannian metric in the plane tailored to the given image, and then computing the corresponding gradient flow. This leads to some new snake models which efficiently attract the given active contour to the features of interest (which basically lie at the bottom of a *potential well*). The method also allows us to naturally write down 3-D active surface models as well. One can completely justify this method using viscosity theory which is done in [15]. Full details of these results can be found in [15]. After this paper was written and submitted for publication, we were informed that a similar method has been reported in [7].

2 Curve Shortening Flows

The motivation for the equations underlying active geometric contours comes from *Euclidean curve short-*

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ening. Therefore, in this section we will review the relevant curve evolution theory in the plane $\mathbf{R}^2$.

Accordingly, for κ the curvature, and $\vec{\mathcal{N}}$ the inward unit normal, one considers families of plane curves evolving according to the *geometric heat equation*

$$\frac{\partial C}{\partial t} = \kappa \vec{\mathcal{N}}. \tag{1}$$

This equation has a number of properties which make it very useful in image processing, and in particular, the basis of a nonlinear scale-space for shape representation [1, 3, 16, 19].

Indeed, (1) is the Euclidean curve shortening flow, in the sense that the Euclidean perimeter shrinks as quickly as possible when the curve evolves according to (1) [11, 13]. Since, we will need a similar argument for the snake model we discuss in the next section, let us work out the details.

Let $C = C(p, t)$ be a smooth family of closed curves where t parametrizes the family and p the given curve, say $0 \le p \le 1$. (Note we assume that $C(0, t) = C(1, t)$ and similarly for the first derivatives.) Consider the length functional

$$L(t) := \int_0^1 \|\frac{\partial C}{\partial p}\| dp.$$

Then differentiating (taking the "first variation"), and using integration by parts, we see that

$$L'(t) \;=\; -\int_0^1 \langle \frac{\partial C}{\partial t}, \frac{1}{\|\frac{\partial C}{\partial p}\|} \frac{\partial}{\partial p}\left[\frac{\frac{\partial C}{\partial p}}{\|\frac{\partial C}{\partial p}\|}\right] \rangle \|\frac{\partial C}{\partial p}\| \rangle dp.$$

But observing now that

$$\|\frac{\partial C}{\partial p}\| dp =: ds$$

is (Euclidean) arc-length, and using the definition of curvature, we see that

$$L'(t) = -\int_0^{L(t)} \langle \frac{\partial C}{\partial t}, \kappa \vec{\mathcal{N}} \rangle ds.$$

Thus the direction in which $L(t)$ is decreasing most rapidly is when

$$\frac{\partial C}{\partial t} = \kappa \vec{\mathcal{N}}.$$

Thus (1) is precisely a gradient flow.

A much deeper fact is that simple closed curves converge to "round" points when evolving according to (1) without developing singularities; see [11, 13]. This fact is one of the keys for the geometric active contour models considered below.

3 2D Active Contour Models

In two elegant papers, Caselles *et al.* [6] and Malladi *et al.* [18] propose a snake model based on the level set formulation of the Euclidean curve shortening equation. More precisely, their model is

$$\frac{\partial \Psi}{\partial t} = \phi(x, y)\|\nabla \Psi\|(\text{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) + \nu). \tag{2}$$

Here the function $\phi(x, y)$ depends on the given image and is used as a "stopping term." For example, the term $\phi(x, y)$ may chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. In [6, 18], the term

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^n} \tag{3}$$

is chosen, where I is the (grey-scale) image and G_σ is a Gaussian (smoothing) filter. (In [6], $n = 1$, and in [18], $n = 2$.) The function $\Psi(x, y, t)$ evolves in (2) according to the associated level set flow for planar curve evolution in the normal direction with speed a function of curvature which was introduced in the fundamental work of Osher-Sethian [21, 22, 25, 26, 27]. It is important to note that as we have seen above, the Euclidean curve shortening part of this evolution, namely $\frac{\partial \Psi}{\partial t} = \|\nabla \Psi\|\text{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right)$ is derived as a gradient flow for shrinking the perimeter as quickly as possible. As is explained in [6], the constant *inflation term* ν is added in (2) in order to keep the evolution moving in the proper direction. Note that we are taking Ψ to be negative in the interior and positive in the exterior of the zero level set contour.

In [18], the inflationary constant is considered both with a positive sign (inward evolution of the evolution of the contour in the direction of decreasing Ψ) and with a negative sign (outward or expanding evolution). (Note the sign convention we have taken for Ψ above.) In the latter case, this can be referred to as expanding "balloons" or "bubbles" as in [28]. For simplicity, unless stated otherwise explicitly, we will take $\nu \ge 0$ (inward evolutions) in what follows below. Instead of using a Gaussian to smooth the image one may of course use the a nonlinear smoothing filter based on the curvature. There are of course many possibilities for a stopping term besides intensity: texture, optical flow, stereo disparity, etc.

We would like to modify the model (2) in a manner suggested by the computation in Section 2. Namely, we will change the ordinary Euclidean arc-length function along a curve $C = (x(p), y(p))^T$ with parameter p given by

$$ds = \|C_p\| dp = (x_p^2 + y_p^2)^{1/2} dp$$

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to

$$ds_\phi = \phi ds = (x_p^2 + y_p^2)^{1/2}\phi dp,$$

where $\phi(x, y)$ is a positive differentiable function. We now essentially repeat the computation made in Section 2, i.e., we want to compute the corresponding gradient flow for shortening length relative to the new metric ds_ϕ.

Accordingly set

$$L_\phi(t) := \int_0^1 \|\frac{\partial C}{\partial p}\|\phi dp.$$

Let

$$\vec{T} := \frac{\partial C}{\partial p}/\|\frac{\partial C}{\partial p}\|,$$

denote the unit tangent. Then taking the first variation of the modified length function L_ϕ, and using integration by parts just as above, we get that

$$L'_\phi(t) = -\int_0^{L_\phi(t)} \langle \frac{\partial C}{\partial t}, \phi\kappa\vec{N} + (\nabla\phi \cdot \vec{T})\vec{T} - \nabla\phi\rangle ds$$

which means that the direction in which the L_ϕ perimeter is shrinking as fast as possible is given by

$$\frac{\partial C}{\partial t} = \phi\kappa\vec{N} + (\nabla\phi \cdot \vec{T})\vec{T} - \nabla\phi. \qquad (4)$$

This is precisely the gradient flow corresponding to the miminization of the length functional L_ϕ. Since the tangential component of equation (4) may be dropped, this may be simplified to

$$\frac{\partial C}{\partial t} = \phi\kappa\vec{N} - \nabla\phi. \qquad (5)$$

The level set version of this is

$$\frac{\partial\Psi}{\partial t} = \phi\|\nabla\Psi\|\text{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right) + \nabla\phi \cdot \nabla\Psi. \qquad (6)$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the *potential well* described by the gradient flow (6). As in [6, 18], we we may also add a constant inflation term (which may be interpreted as a Lagrange multiplier for a constrained version of the given optimization problem), and so derive a modified model of (2) given by

$$\frac{\partial\Psi}{\partial t} = \phi\|\nabla\Psi\|(\text{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right) + \nu) + \nabla\phi \cdot \nabla\Psi. \qquad (7)$$

Notice that for ϕ as in (3), $\nabla\phi$ will look like a doublet near an edge. Of course, one may choose other candidates for ϕ in order to pick out other features.

We have implemented this snake model based on the algorithms of Osher-Sethian [21, 22, 25, 26, 27] and Malladi *et al.* [18]. Some preliminary numerical results with our code will be presented in below.

Note that the metric ds_ϕ has the property that it becomes small where ϕ is small and vice versa. Thus at such points lengths decrease and so one needs less "energy" in order to move. Consequently, it seems that such a metric is natural for attracting the deformable contour to an edge when ϕ has the form (3).

4 3-D Active Contour Models

In this section, we will discuss some possible geometric 3-D contour models based on surface evolution ideas, by modifying the Euclidean area in this case by a function which depends on the salient features which we wish to capture. In order to do this, we will need to set up some notation.

Let $S : [0, 1] \times [0, 1] \to \mathbf{R}^3$ denote a compact embedded surface with (local) coordinates (u, v). Let H denote the mean curvature and $\vec{N}$ the inward unit normal. We set

$$S_u := \frac{\partial S}{\partial u}, \quad S_v := \frac{\partial S}{\partial v}.$$

Then the infinitesimal area on S is given by

$$dS = (\|S_u\|^2\|S_v\|^2 - \langle S_u, S_v\rangle^2)^{1/2}dudv.$$

Let $\phi : \Omega \to \mathbf{R}$ be a positive differentiable function defined on some open subset of $\mathbf{R}^3$. The function $\phi(x, y, z)$ will play the role of the "stopping" function ϕ given above in our snake model (6, 7).

It is a beautiful classical fact that the gradient flow associated to the area functional for surfaces (i.e., the direction in which area is shrinking most rapidly) is given by

$$\frac{\partial S}{\partial t} = H\vec{N}. \qquad (8)$$

What we propose to do is to replace the Euclidean area by a modified area depending on ϕ namely,

$$dS_\phi := \phi dS.$$

For a family of surfaces (with parameter t), consider the ϕ-area functional

$$A_\phi(t) := \int\int_S dS_\phi.$$

Once again, an integration by parts argument shows that

$$\frac{\partial S}{\partial t} = \phi H\vec{N} - \nabla\phi, \qquad (9)$$

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gives the direction in which $A_\phi(t)$ is shrinking most rapidly. The level set version of (9) is given in terms of $\Psi(x, y, z, t)$ by

$$\Psi_t = \phi \|\nabla\Psi\| \mathrm{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right) + \nabla\phi \cdot \nabla\Psi. \qquad (10)$$

As before one may add a constant inflation term to the mean curvature to derive the model

$$\Psi_t = \phi \|\nabla\Psi\| \left(\mathrm{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right) + \nu\right) + \nabla\phi \cdot \nabla\Psi. \qquad (11)$$

In the context of image processing, the term ϕ depends on the given 3-D image and is exactly analogous to the stopping term in (6, 7). It is important to note that there is a very big difference between the 2-D and 3-D models discussed here. Indeed, the geometric heat equation will shrink a simple closed curve to a round point without developing singularities, even if the initial curve is *nonconvex*. The geometric model (2) is based on this flow. For surfaces, it is well-known that singularities may develop in the mean curvature flow (8) non-convex smooth surfaces [12]. (The classical example is the dumbbell.) We should note however that the mean curvature flow does indeed shrink smooth compact convex surfaces to round "spherical" points.

We should add that because of these problems, several researchers have proposed replacing mean curvature flow by flows which depend on the Gaussian curvature κ. Indeed, define

$$\kappa_+ := \max\{\kappa, 0\}.$$

Then Caselles and Sbert [8] have shown that the *affine invariant flow*

$$\frac{\partial S}{\partial t} = \mathrm{sign}(H)\kappa_+^{1/4}\vec{\mathcal{N}} \qquad (12)$$

will (smoothly) shrink rotationally symmetric compact surfaces to ellipsoidal shaped points. (This has been proven in [4] in the convex case. See also [2].) Thus one could replace the mean curvature part by $\mathrm{sign}(H)\kappa_+^{1/4}$ in (11). Another possibility would be to use $\kappa_+^{1/2}$ as has been proposed in [20]. See also [28]. (Note that Chow [9] has shown that convex surfaces flowing under $\kappa^{1/2}$ shrink to spherical points.) All these possible evolutions for 3-D contours are now being explored. See also Section 5 below for a possible aternative related approach to 3D segmentation.

5 Geodesic Paths

Given that we are now looking at a Riemannian metric, it becomes natural to investigate its geodesics.

It turns out that the geodesics appear to have a very concrete interpretation in this problem. For background on the problem of finding closed geodesics on manifolds in connection with curve-shortening, see [10, 11, 13].

It is therefore natural to investigate the geodesics of the conformally Euclidian metric considered in this paper; this will lead to an *ordinary differential equation* which can be used both for the segmentation and edge detection problem.

We work in the plane once again with the stopping function ϕ as discussed in Section 3. As before we modify the Euclidean metric by taking $ds_\phi = \phi ds$. Indeed, setting $u = \phi^2$, we find that the equation of geodesics takes the form [10]

$$C_{pp} + \frac{1}{u}[\langle C_p, \nabla u\rangle C_p - \frac{1}{2}\|C_p\|^2\nabla u] = 0. \qquad (13)$$

Note that the tangential term $\langle C_p, \nabla u\rangle C_p$ can be removed by change of parametrization.

Define the "potential function"

$$V(C) = E - \frac{\phi^2}{2},$$

where E is a certain constant which we describe below. Given this we claim that solutions of the oscillator-type equation

$$C_{qq} + \nabla V(C) = 0 \qquad (14)$$

can be reparametrized to give geodesics of the line element ds_ϕ. This may be seen from the following calculation. Start with the equation (13). Since we have the first integral

$$\frac{1}{2}\|C_q\|^2 + V(C) = E,$$

equation (14) can be written as

$$C_{qq} + \frac{\|C_q\|^2}{2(E - V)}\nabla V = 0, \qquad (15)$$

which is precisely (13) up to tangential term.

In terms of the image, what we want is to locate a potential valley. Therefore, an algorithm using geodesics might be as follows: run a geodesic with zero initial speed, and stop when its velocity starts to decrease. This should approximately give a point on the desired contour. Of course, one may want to add a stopping term too. Also, such a geodesic, if launched in the correct direction, should tend to *go around* the desired object—this might be useful to get an initial guess, and for segmentation.

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We should note that the above computation remains valid in any number of space dimensions. For surfaces, these curves would swirl about the desired object. Since the equations are conservative, we are guaranteed that the curves will not wander too far away. Numerical calculations using this idea will be reported elsewhere.

6 Numerical Experiments

The implementations we have used in this paper are based on the level set evolution methods developed by Osher-Sethian [21, 22, 25, 26, 27], and the techniques in [18].

We have been applying our methods to a number of medical images. For purposes of illustration, we will present two such images here. The first is an MRI noisy myocardial image. We have captured the chambers using an outward evolution (the "balloon" or "bubble" technique). The second image is an ultrasound image of a breast cyst whose contour we also find using our curve evolution methods.

7 Conclusions

In this paper, we presented a new snake methodology derived by considering the curve shortening evolution relative to a conformally Euclidean metric. Thus the evolution is based on a "natural" energy function which is intimately related to the geometry of the image. We are now in the process of extending these results in order to develop 3D segmentation algorithms.

Acknowledgements

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