FEATURE-PRESERVING FLOWS: A STOCHASTIC DIFFERENTIAL EQUATION'S VIEW

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ABSTRACT

Evolution equations have proven to be useful in tracking fine to coarse features in a single level curve and/or in an image. In this paper, we give a stochastic insight to a specific evolution equation, namely the geometric heat equation, and subsequently use this insight to develop a class of feature-driven diffusions. A progressive smoothing along desired features of a level curve is aimed at overcoming effects of noisy environment during feature extraction and denoising applications.

1. INTRODUCTION

Feature extraction is one of the most important areas of image analysis and computer vision. In the course of most image processing operations, for instance denoising or restoration, the aim is to preserve and extract features to: (i) discriminate among objects, (ii) track moving objects. Some widely used features in an image are edges, corners, vertices, color, blob-like and ridge-like structures. For industrial applications such as one involving a conveyor belt in a factory for instance, one may want to identify corners of passing objects and to discriminate between triangles, rectangles, circles etc. In applications of tracking moving objects, this may be achieved by tracking some other specific features of target objects. For example, chromaticity of an object can be tracked, or so can be its corners.

In recent years evolving level sets of two-dimensional functions or images in time/scale via a Partial Differential Equation (PDE) has emerged as an important tool in image processing. The usefulness of these evolutions comes from the fact that they provide a scale-space analysis where fine to coarse features and properties of a signal can be fully traced and observed. The goal of feature and shape extraction in recognition and classification problems has long been hampered by noise and processing artifacts. The idea of feature-driven progressive smoothing and scale tracking is widely viewed as a promising new avenue of research, and hence has been of increasing interest to researchers in the field.

In this paper, we provide a stochastic insight to a specific evolution equation, namely the geometric heat equation, and subsequently use this insight to propose a class of nonlinear diffusions aimed at emphasizing desired features in a possibly noisy environment. (See [1, 2] for a stochastic approach taken to address a different problem.) Following a brief review of some background material in the next section, we formulate a stochastic interpretation of the geometric heat flow in Section 3, and in Section 4 we describe our new approach to feature-preserving smoothing along with illustrating examples.

2. BACKGROUND

Almost all denoising/restoration methods together with feature tracking methods have long relied on lowpass filtering of an initial signal $u_o(x,y)$, by a Gaussian filter $G_\sigma(x,y)$ with variance $\sigma^2 = 2kt$, $u(t,x,y) = u_o(x,y) \star G_\sigma(x,y)$. Witkin [3] noted that convolving a function with a Gaussian filter of variance $2kt$ was equivalent to a linear diffusion equation, which may be written as $(u(0,x,y) = u_o(x,y))$:

$$\frac{\partial u(t,x,y)}{\partial t} = k\Delta u(t,x,y).$$

The obvious drawback with a linear diffusion approach to feature detection is that a significant amount of blurring is introduced. This led researchers in this field to seek nonlinear diffusion methods which would denoise an image as well as keep its desired features such as edges and corners.

To avoid the oversmoothing effect of a linear diffusion and to help preserve edges, the so-called anisotropic diffusion was proposed by Perona-Malik [4]. The idea was to slow down diffusion at high gradient locations, i.e., at edges which are the features to be preserved in the image: $\frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\ \nabla u)$. The initial signal $u_o(x,y)$ may also be viewed as a collection of iso-intensity contours, or a level set function, where the normal and tangent directions are respectively denoted by η and ξ. Since the Laplacian is a rotationally invariant-operator, it may also be expressed

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as $\Delta u = u_{\xi\xi} + u_{\eta\eta}$. Constraining the smoothing to the tangent direction ξ results in what is referred to as Geometric Heat Eqn. (GHE) which in terms of Euclidean coordinates, is expressed as

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x,y) &= u_{\xi\xi}(t,x,y) = A_{GHE}\, u(t,x,y) \\ &= \frac{u_y^2\, u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \\ u(0,x,y) &= u_o(x,y),\end{aligned}$$

which may be shown to be equivalent to evolving each of the level curves of $u_o(x,y)$ by curvature, as done with parametric forms of curves in differential geometry (Euclidean shortening flow).

3. GHE AND STOCHASTIC DIFFERENTIAL EQUATIONS

Defining the normal angle of a level curve in terms of $u(x,y)$: $\theta(t,x,y) = \tan^{-1}\left(\frac{u_y(t,x,y)}{u_x(t,x,y)}\right)$, the geometric heat operator $\mathbf{A}_{GHE}$ may be written as

$$\begin{aligned}\mathbf{A}_{GHE}\, u(t,x,y) &= \sin^2(\theta(t,u_x,u_y))\, u_{xx}(t,x,y) \\ &-2\sin(\theta(t,u_x,u_y))\cos(\theta(t,u_x,u_y)) u_{xy}(t,x,y) \\ &+ \cos^2(\theta(t,u_x,u_y))\, u_{yy}(t,x,y). \qquad (1)\end{aligned}$$

It is known that a 2^{nd} order partial differential operator can be associated to an Ito diffusion X_t as the generator of the process [5], referred to as the infinitesimal operator. An Ito diffusion X_t is a mathematical model for the trajectories in time X_t of a small particle suspended in a moving liquid, with a corresponding Stochastic Differential Equation (SDE) given by

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t,$$

where b and σ are respectively drift and diffusion coefficients, and are determined from the generator, and B_t is a standard Brownian Motion (BM) [5].

A natural question which arises in light of the geometric heat flow operator $\mathbf{A}_{GHE}$, is about an underlying diffusion with its corresponding SDE, when given a solution $u(t,x,y)$ of the PDE at scale t. We can show [6] that this leads to the following SDE:

$$\begin{aligned}dX_t &= \sqrt{2}\begin{pmatrix} -\sin(\theta(t,u_x,u_y)) \\ \cos(\theta(t,u_x,u_y))\end{pmatrix} dB_t \\ &= \sqrt{2}\ \vec{T}(t,u_x,u_y)\ dB_t. \qquad (2)\end{aligned}$$

This SDE explains that GHE generates a Brownian Motion of particles on a very local tangential frame of each pixel, and since BM is an averaging process, iso-intensity contours are smoothed maximally. In the next section, we explain this new interpretation further, and use the resulting intuition for developing novel feature-preserving flows.

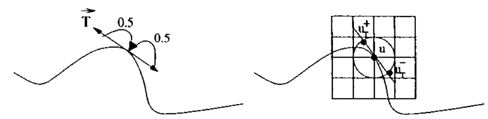

Fig. 1. Brownian Motion on the tangent direction, and corresponding interpolation on square grid.

Note that the diffusion coefficient in Eq. (2) is the tangent vector of $u(t,x,y)$ at each position, which depends on the solution of the PDE, $u(\cdot,\cdot)$, and its derivatives, u_x and u_y at each time step. If $u(t,x,y)$ and its derivatives are "sufficiently regular" (Lipschitz properties), for each time step δt, a diffusion X_t is constructed, and may be used to write a Backward Kolmogorov Equation,

$$u(t-\delta t, x) = E\{u(t,X_t)/X_{t-\delta t} = x\},$$

as a mean value around each pixel dictated by the motion of the constructed diffusion X_t. This equation can also be written in forward time to give way to an averaging process in the tangent direction carried out during a forward evolution (i.e. estimate the new pixel value at time t as a mean value of two neighbor pixel values on the tangent at time $t - \delta t$).

Equation (2) then amounts to a pixel displacement along the tangent of a curve -and according to a Brownian motion- on a discrete lattice such a motion is captured by a random walk with equally likely (i.e., prob. 1/2) displacements to u_T^+ and u_T^-. The latters are obtained by a bilinear interpolation around u (in x and y direction and along the tangent, See Fig. 1). As a result, we can write such an equation as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\Delta_T\, u \simeq \frac{u_T^+ - u}{2} + \frac{u_T^- - u}{2},$$

where Δ_T is the Laplacian operator in tangent direction $\vec{T}$.

For simulation purposes, we use a level set methodology [7], which has an advantage of naturally handling topology changes on the level set function in Eulerian framework. A simulation example where a "T" shape is evolved via a discrete equivalent of BM, i.e., the random walk, in a tangent direction, is shown in Fig. 2.

4. TOWARDS FEATURE-PRESERVING FLOWS

As it might be observed from the simulation example shown in the previous section, contours subjected to a GHE flow will lose all important features, particularly corners, and will become circular.

Using the stochastic interpretation of the GHE, it follows that some specific features might be desired to remain invariant in the course of smoothing, e.g. corners. One possible solution is to detect the presence of such features (i.e.

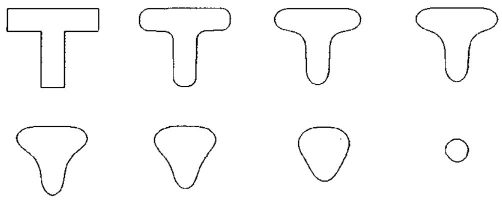

Fig. 2. Equivalent random walk on the tangent direction implemented on the level set function $u(x,y)$. The tangent direction is estimated directly from the level set set function : $\theta_T = \tan^{-1}\left(\frac{-u_x}{u_y}\right)$ The level set function is on a 250×250 grid, $\delta t = 0.25$.

corners) by examining the rate of change of the normal angle θ. Any large variation between u_T^+ and u_T^- signifies a significant corner which should, in turn, slow down the diffusion at that point for better preservation as exhibited in Fig. 3. In a region of small change in θ on the other hand, it is more likely for there to be a straight line, and hence allow for a tangential diffusion to proceed, and thereby eliminate oscillations likely due to noise. Inspired by Perona's remarks that finite differences of angles should take place on a circle [8], we are led to construct a weighting function $e^{-5\sin^2(\nabla\theta)}$, where $\nabla\theta$ is as shown in Fig. 3. As a result, we propose the following feature preserving flows with $p^+ = e^{-5sin^2(\theta^+ - \theta)}$, and $p^- = e^{-5sin^2(\theta^- - \theta)}$;

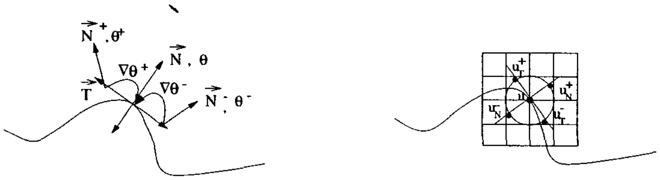

Fig. 3. BM on $\vec{T}$ considering orientation changes, and corresponding interpolations on a square grid to find neighbors on tangent and normal directions of a level curve.

$$\frac{\partial u}{\partial t} = p^+\sqrt{p^-}\frac{(u_T^+ - u)}{2} + \sqrt{p^+}p^-\frac{(u_T^- - u)}{2}, \quad (3)$$

$$\frac{\partial u}{\partial t} = p^+\sqrt{p^-}\frac{(u_T^+ - u)}{4} + \sqrt{p^+}p^-\frac{(u_T^- - u)}{4} + (p^+p^-)^{3/2}\frac{(u_N^+ - 2u + u_N^-)}{4}. \quad (4)$$

In Eq. (3), we use only neighbors on the tangent, and weight the Brownian motion of particles with weights $p^+\sqrt{p^-}$ on the +-labeled side, and vice-versa. The idea is that we want to limit the motion when not only one but both weights are small and to give more weight to the corresponding side's information as well. In Eq. (4), we incorporate the normal neighbors' information into the nonlinear averaging process. We weight the diffusion in this direction by $(p^+p^-)^{3/2}$, so that when both of the weights are significantly "large" we get a contribution from this movement with the hope of some enhancement of the corner, which points in this direction.

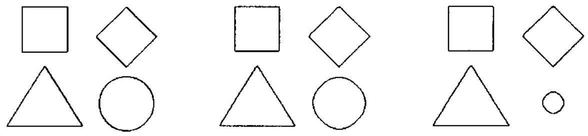

Fig. 4. Random walk on the tangent direction weighted by a function of $|\nabla\theta|$ with neighbors on $\vec{T}$. Left: Initial Shapes, Middle: scale $t = 175$, Right: t=1250.

In Fig. 4, we demonstrate such a technique. Starting with clean shapes, and applying the proposed flow (Eq. (4)), it can be observed that we accomplish our goal of feature-preservation since all shapes but a circle remain intact. The form of the weighting function determines the range of angle turns to be kept (see Fig. 5). Here, acute angles of the triangle, right angles of the square and of the diamond remained invariant, whereas very small angle turns on a small tangential neighborhood of the circle allowed continued diffusion of the circle. Example in Fig. 6 shows the output of

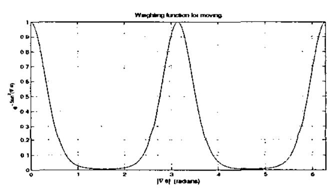

Fig. 5. Function weighting change in normal angles.

the feature-preserving flow as well as GHE acting on a noisy shape. For getting reliable estimates of the tangent and normal angles at each point, we apply our flow (Eq. (4)) to a slightly smoothed version of the noisy shape, i.e. start with the shape at scale $t = 2.5$. The important features of the plane shape can be observed in the output where noise is also removed. On the other hand, continued application of GHE rounded features of the plane. We show the nice property of the proposed flows in Fig. 7 by demonstrating their performance on both clean and noisy versions of a star shape.

5. DISCUSSION

What we proposed in the previous section is an improvement on our previous work [6], where we used the insight of working with the normal angle θ, and proposed the idea of constraining BM at some pre-determined orientation angles ($u_t = h^2(\theta)u_{\xi\xi}$). Our goal was to focus on preserving

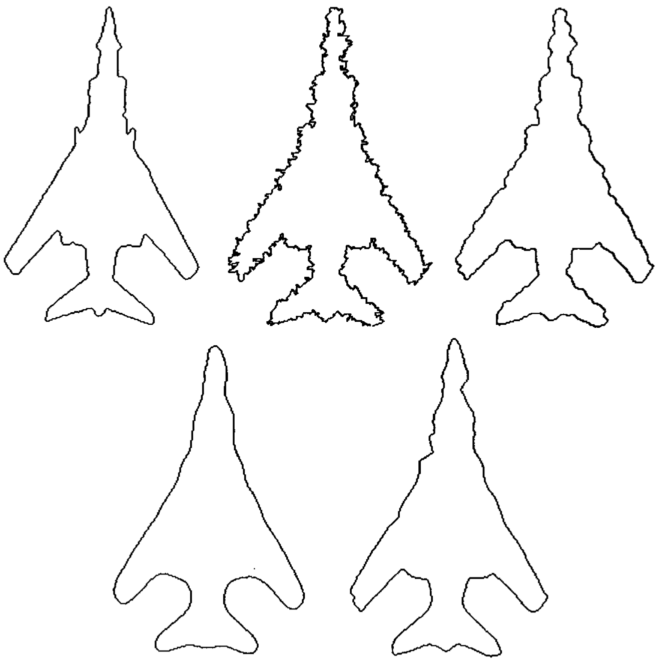

Fig. 6. Left-to-Right: (Top) Actual Plane Shape, Noisy Plane Shape, GHE flow t=2.5; (Bottom) GHE flow t=25, Proposed flow (Eq. (4)) t=25.

specific shapes, and we showed that the proposed flows led level curves to polygons when we chose periodic functions ($h^2(\theta) = \sin^2(n\theta)$ or $\cos^2(n\theta)$, $n \in \mathbb{R}$), whose periodicity assumes the desired number of vertices. For images, these shape-adapted flows are aimed to drive level curves towards a specific shape so that such features are enhanced. (See Fig.8) In order to lift the requirement of a prior information of alignment of desired features, we used $\nabla\theta$ instead of θ in the newly proposed flows. This helped us obtain an autonomously feature-driven smoothing of curves.

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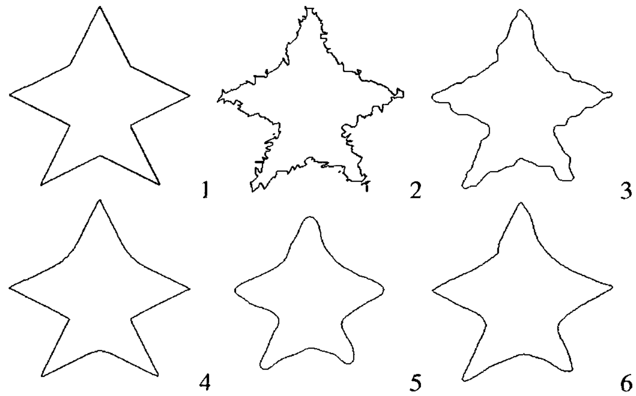

Fig. 7. (Top) 1. Actual Star Shape 2. Noisy Star Shape 3. GHE t=7.5 (Bottom) 4. Our flow (initial shape in 1) t=125 5. GHE t=37.5 6. Our flow (initial shape in 3) t=125.

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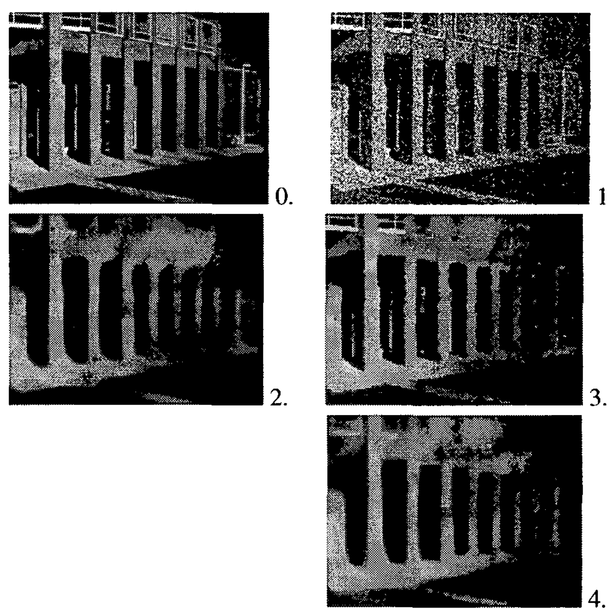

Fig. 8. 0. Actual building image 1. noisy building image 2. GHE $u_t = u_{\xi\xi}$, $t = 10$. 3. P-M flow[4], $t = 40$. 4. Flow $u_t = \cos^2(2\theta)\, u_{\xi\xi}$, $t = 40$.