31° Convegno Nazionale di Idraulica e Costruzioni Idrauliche
Perugia, 9-12 settembre 2008

EULER CHARACTERISTICS & MAXIMA OF OCEANIC SEA STATES VIA A VARIATIONAL WAVE ACQUISITION STEREO SYSTEM

F. Fedele[1], G. Gallego[2], A. Benetazzo[3] & A. Yezzi[2]

(1) School of Civil & Environmental Engineering, Georgia Institute of Technology, Savannah campus, USA
(2) Protecno S.r.l , Padua ITALY
(3) School of Electrical & Computer Engineering, Georgia Institute of Technology, Atlanta campus, USA

ABSTRACT

We propose a novel Variational Wave Acquisition Stereo System (VWASS) that exploits new stereo reconstruction techniques for accurate estimates of the space-time dynamics of oceanic sea states. The rich information content of the acquired three-dimensional video data is used to make predictions on directional spectra and large waves over an area. To do so, we present a statistical analysis of the Euler Characteristic (EC) of excursion sets of the wave surface reconstructed via VWASS. These results have broader impact in offshore engineering for more accurate predictions of large waves in sea storms passing by offshore structures.

1 INTRODUCTION

The prediction of large waves is typically based on the statistical analysis of time series of the wave surface displacement retrieved from wave gauges, ultrasonic instruments or buoys. However, the largest wave crest predicted at a fixed point P underestimates the highest wave expected over an area. Indeed, because large waves travel on top of wave groups, the probability that the group passes at its apex through P is practically null. The largest crest recorded in time at P is simply due to the dynamical effects of a group that will focus nearby that location. Can we predict the largest wave expected over a given area?

In this paper, we address this question by proposing a novel variational Wave Acquisition Stereo System (VWASS) for the reconstruction of the water surface. The rich information content of the acquired three-dimensional video data is then exploited to compute reliable estimates of the expected global maximum (largest crest height) over an area using the Euler Characteristic of excursion sets (Adler 1981, Taylor & Adler 2007).

The paper is structured as follows. We first discuss the mathematical formulation of VWASS and how it is used in applications. We then introduce the concept of the Euler Characteristic (EC) of the h-excursion set (Adler, 1981)

$$A_{\eta,h} = \{(x,y) \in S : \eta(x,y) > h\} \qquad (1)$$

of spatial snapshots η of oceanic sea states over an area S, and its relation to the expected number of maxima and *h*-upcrossings. We then present estimates from video data of both directional wave spectra and empirical exceedance probabilities of the global maximum of η over *S*. The broader impact of these results to offshore engineering is finally discussed.

2 THE STEREO VARIATIONAL GEOMETRIC METHOD

The reconstruction of the wave surface from stereo pairs of ocean wave images is a classical problem in computer vision commonly known as the correspondence problem (Ma et al. 2004). Its solution is based on epipolar geometry techniques that find corresponding points in the two images, from which one obtains the estimate of the real point in the three dimensional terrestrial coordinate system. The Wave Acquisition Stereo System (WASS) developed by Benetazzo (2006) utilizes the 'epipolar algorithm' for the space-time reconstruction of the sea surface. However, this approach may fail to provide a smooth surface reconstruction because of "holes" corresponding to unmatched image regions (Ma et al. 2004, Benetazzo 2006). For example, this can occur when, at a given point on the water surface, the same amount of light is received from all possible directions and reflected towards the observer causing a visual blurring of the specularities of the water. This is typical of cloudy days, and the water surface is said to support a Lambertian radiance function (Ma et al. 2004). We address this problem by proposing a novel formulation of WASS based on variational principles (VWASS). Under the assumptions of a Lambertian surface, following the seminal work by (Faugeras et al. 1998), the 3-D reconstruction of the water surface is obtained in the context of active surfaces by evolving an initial surface through a PDE derived from the gradient descent flow of a cost functional designed for the stereo reconstruction problem.

To be more specific, the energy being *maximized* is the normalized cross correlation between the image intensities obtained by projecting the same water surface patch onto both image planes of the cameras. It is clear that such energy depends on the shape of the water surface. Therefore, the active surface establishes an evolving correspondence between the pixels in both images. Hence, the correspondence will be obtained by evolving a surface in 3-D instead of just performing image-to-image intensity comparisons without an explicit 3-D model of the target surface being reconstructed.

To infer the shape of the water surface $\eta(x,y)$ at the location (x,y) over an area *S*, we set up a cost functional on the discrepancy between the projection of the model surface and the image measurements. As previously announced, such cost is based on a cross correlation measure between image intensities, which will be noted as $E_{\mathrm{data}}(\eta)$. We conjecture that, to have a well-posed problem, a regularization term that imposes a geometric prior must also be included, $E_{\mathrm{geom}}(\eta)$. We consider the cost functional to be the (weighted) sum:

$$E(\eta) = E_{\mathrm{data}}(\eta) + E_{\mathrm{geom}}(\eta). \tag{2}$$

In particular, the geometric term favors surfaces of least area:

$$E_{\mathrm{geom}}(\eta) = \int_{\eta} dA\,. \tag{3}$$

The data fidelity term may be expressed as

$$E_{data}(\eta) = \int_{\eta} \left(1 - \frac{\langle I_1, I_2 \rangle}{|I_1| \cdot |I_2|} \right) dA, \quad (4)$$

where η is the wave surface region within the field of views of both cameras, and $\langle I_1, I_2 \rangle$ is the cross-correlation between the image intensities I_1 and I_2 .

To find the surface η that minimizes *E*, we start from an initial estimate of the surface at time $t = 0$, η_0, and set up a gradient flow based on the first variation of *E* that will make the surface evolve towards a minimizer of *E*, hopefully converging to the desired water surface shape.

Based on the theorem in (Faugeras et al. 1998) that says that for a function $\Phi : R^3 \times R^3 \rightarrow R^+$ and the energy

$$E = \int_{\eta} \Phi(X, N) dA, \quad (5)$$

where *N* is the unit normal to η at *X*, the flow that minimizes *E* is given by the evolution PDE

$$\eta_t = \beta N, \quad (6)$$

where η_t is the derivative of η with respect to a fictitious time variable and the speed β in the normal direction to the surface that drives the evolution is

$$\beta = 2H(\Phi - \Phi_N \cdot N) - \Phi_X \cdot N - trace\left[(\Phi_{XN})_{T_\eta} + dN \circ (\Phi_{NN})_{T_\eta} \right] \quad (7)$$

All quantities are evaluated at the point η =*X* with normal *N* to the surface. *H* denotes the mean curvature. Φ_X, Φ_N are the first-order derivatives of Φ , while Φ_{XN}, Φ_{NN} are the second-order derivatives. dN is the differential of the Gauss map of the surface and $(\cdot)_{T_\eta}$ means "restriction to the tangent plane T_η to the surface at η=*X*". Note that our proposed energy (1) can be expressed in the form of (4) if $\Phi = \left(1 - \langle I_1, I_2 \rangle / |I_1| \cdot |I_2| \right) + \alpha$, where α is just a weight for the geometric prior. In practice, we use the flow based on the first-order derivatives of Φ because it provides similar results to those of the complete expression, but saves a significant amount of computations,

$$\eta_t = \left(2H(\Phi - \Phi_N \cdot N) - \Phi_X \cdot N \right) N . \quad (8)$$

The level set framework has been adopted to numerically implement (6). For the technical description of the variational stereo algorithm implementation we refer to Gallego et al. (2008). We have tested the variational reconstruction algorithm using a set of images, shown in Fig. 1, acquired by Benetazzo (2006) on a water depth of 8 meters. Fig. 2 shows the successful reconstructed surface and the associated directional wave spectrum. Hereafter, we introduce the concept of the Euler characteristic (EC) that will be applied to predict the expected number of large maxima in oceanic sea states exploiting the high statistical content of the acquired video data via VWASS.

3 EULER CHARACTERISTIC OF EXCURSION SETS AND MAXIMA OF RANDOM FIELDS

In algebraic topology, the **Euler characteristic** *EC* is classically defined for polyhedra, according to the formula

$$EC = V - E + F \quad (9)$$

where *V*, *E*, and *F* are respectively the numbers of vertices, edges and faces in the given polyhedron. The same definition given in (9) also holds for two dimensional (2D) sets which are the focus of this paper. In this case, the *EC* of a 2D set is also equivalent to

$$EC = \#\text{'connected components'} - \#\text{'holes'}. \tag{10}$$

For example, for a square, *EC*=4-4+1=1 according to (9), which is in agreement with (10) since there is only 1 connected component and no holes.

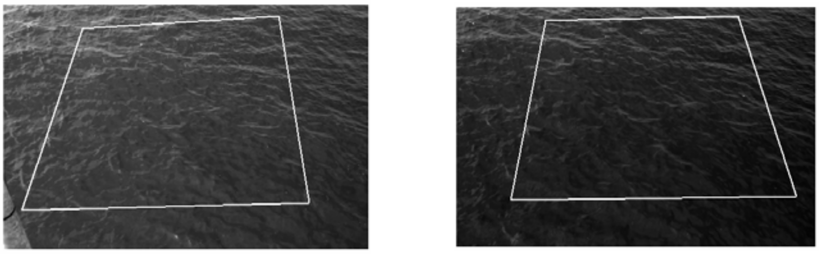

Figure 1: Input stereo pair images to the algorithm. The rectangular domain (8 m × 8.7 m) of the reconstructed surface or elevation map (right column) has been superimposed. The height of the waves is in the range ±0.2 cm.

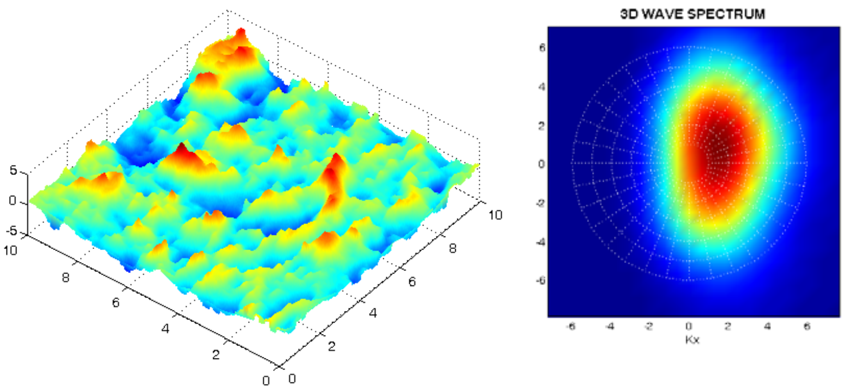

Figure 2: (left) Reconstructed normalized wave surface η via VWASS; (right) estimate directional wave spectrum of η.

For a generic 2D set *B*, computing the *EC* from the definition (10) presents some challenges. A computationally efficient approach can be devised based on (9). First, define a Cartesian mesh grid Γ of size (Δx, Δy) that approximates the complicated domain of the given set *B*. The *EC*(Γ) is then given by the sum of the *EC* of all the inner squares composing Γ. In particular, denote F as the number of squares (faces) composing Γ, E_h (E_v) as the number of horizontal (vertical) segments between two neighboring mesh points and V the number of points. The EC of the set then follows from (9) setting $E = E_h + E_v$. As the grid cell size ΔxΔy tends to zero, we expect *EC*(Γ) to tend to the Euler characteristic of the continuous set *B*.

As an example, consider the z-excursion set $A_{Z,z}$, for z=1, of a Gaussian field *Z* defined illustrated in Fig. 3.

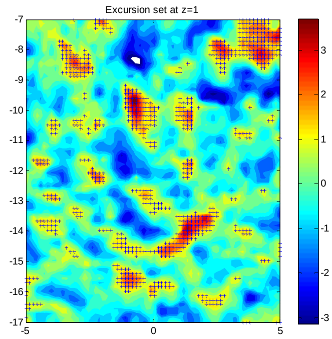

Figure 3: Excursion set $A_{Z,z}$ for *z*=1 of a Gaussian field with mean zero and unit variance and superimposed mesh grid Γ which approximates $A_{Z,1}$.

We recall that the *z*-excursion set $A_{Z,z}$ is the sub-region where the random surface *Z* exceeds a fixed threshold *z*. The mesh grid Γ approximating the set is also superimposed on $A_{Z,1}$. In this case, *EC*(Γ)=20 since $A_{Z,1}$ is mainly a collection of connected components in agreement with (10).

In general, the *EC* of an excursion set of a 2D random field depends very strongly on the threshold *z*. If this is low, then *EC* counts the number of holes in the given set. If the threshold is high, then all the holes tend to disappear and the *EC* counts the number of connected components, or local maxima of the random field *Z*. For a stationary Gaussian field η, an exact formula for the expected value of *EC*, valid for any threshold, was discovered by Adler (1981). For 2D Gaussian fields this is given by

$$EC(A_{\eta,h}) = A_S \,(2\pi)^{-3/2} |\mathbf{\Lambda}|^{1/2} \xi e^{-\xi^2/2}, \tag{11}$$

where $\xi = h/\sigma$ is the normalized threshold amplitude, A_S is the area of region *S*, and $\mathbf{\Lambda}$ is the covariance matrix of the gradient $\nabla\eta$. If the excursion set touches the boundary of the area *S* special boundary corrections need to be added (Worsley 1995), but hereafter these terms will be neglected without any lost of accuracy in the final results. Why the *EC* of an excursion set is relevant for engineering applications?

Adler & Taylor (2007) have shown that the probability that the global maximum over an area *S* exceeds a threshold *h* is well approximated by the expected *EC* of the excursion set $A_{\eta,h}$, provided the threshold is high. That is,

$$\Pr\left(\max_{P\in S} \eta(P) > h\right) \approx EC(A_{\eta,h}), \quad for\ h >> 1. \tag{12}$$

A heuristic explanation of why this result holds is as follows: as the threshold *h* increases, the holes in the excursion set $A_{\eta,h}$ disappear until each of its connected components includes just one local maximum. At this point, the *EC* counts the number of local maxima. Raising the threshold still further, until only the global maximum remains, the *EC* takes the value 1 if the global maximum exceeds the threshold and 0 if it is below. A consequence of (12) is that, for h>>1

$$EX_{\max}(h) \approx EC(A_{\eta,h}), \tag{13}$$

that is, the expected number $EX_{\max}(h)$ of large local maxima equals the *EC* of large excursion sets. We note that in a one-dimensional (1D) random process, $EC(A_{\eta,h})$ is the expected number of h-upcrossings. Thus, in 1D processes (13) simply states that the expected number of large local maxima equals that of large h-upcrossings, implying the well known one-to-one correspondence between h-upcrossings and maxima at large thresholds. In general, for two dimensional (2D) random fields this correspondence does not hold since up-crossings are contour levels. Can we define an appropriate h-upcrossing such that the correspondence with maxima of 2D random fields is also one-to-one?

The answer to this question follows from the seminal work of Adler (1976) on generalizing upcrossings to higher dimensions. For simplicity, consider a 2D Gaussian field η on a cartesian coordinate system (t,s) for which the covariance matrix $\mathbf{\Lambda}$ of $\nabla\eta$ is diagonal with spectral moments m_{tt} and m_{ss}, where $m_{tt} > m_{ss}$. Note that the t-axis is along the principal direction θ (with respect to the original x axis) where the moment m_{tt} of the wave number spectrum of η along θ attains its maximum. In this setting, the partial derivatives $\partial_t\eta$ and $\partial_s\eta$ of η are uncorrelated, and stochastically independent. With this setting in mind, an h-upcrossing occurs at a point $P \in S$ if

i) a h-upcrossing of η occurs along t ($\eta = h, \partial_t\eta > 0$ at P) and

ii) η attains a local maximum along s, i.e. η is convex along s ($\partial_s\eta = 0, \partial_{ss}\eta < 0$ at P).

Note that the extra condition (ii) is necessary to guaranty that, locally at P, η is rising. Further, this definition does not depend on the particular choice of the coordinate axes, and for large thresholds h, each 2D upcrossing corresponds uniquely to a large local maximum of η. Indeed, following Rice logic (Adler 1981), the expected number $EX_+(h)$ of h-upcrossings is given by the following generalized Rice formula

$$EX_+(h) = A_S \int_{w_1=0}^{\infty} \int_{w_2=-\infty}^{0} w_1 |w_2| p(\eta = h, \partial_t\eta = w_1, \partial_s\eta = 0, \partial_{ss}\eta = w_2) dw_1 dw_2 \tag{14}$$

where $p(\bullet)$ is the joint probability density function (pdf) of $\eta, \partial_t\eta, \partial_s\eta, \partial_{ss}\eta$. For an exact solution of (14) we refer to Adler (1981). Instead, the asymptotic solution of (14) for h>>1 can be derived as follows. By Gaussianity, $\partial_t\eta$ and $\partial_s\eta$ are independent from $\partial_{ss}\eta$ and η, and it follows that

$$EX_+(h) = A_S \int_{w_1=0}^{\infty} w_1 p(\partial_t\eta = w_1) dw_1 \cdot \left[p(\partial_s\eta = 0) \int_{w_2=-\infty}^{0} |w_2| p(\eta = h, \partial_{ss}\eta = w_2) dw_2 \right].$$

The first integral on the left is equal to $\sqrt{m_{tt}/2\pi}$, whereas the term within square brackets equals the expected number of local maxima along s, per unit length with amplitude *h*, which is given, for large *h*, by $\sqrt{m_{ss}/2\pi}\,\xi\exp(-\xi^2/2)$, with $\xi = h/\sigma$. Noting that the determinant $|\Lambda| = m_{tt}m_{ss}$ is invariant by any axes rotation, we finally conclude that, for large thresholds, $EX_+(h)$ equals $EC(A_{\eta,h})$ of (11), implying that for *h*>>1

$$EX_+(h) \approx EX_{\max}(h) \approx EC(A_{\eta,h}).$$

This proves the existence of a one-to-one correspondence between 2D upcrossings and large maxima as in 1D processes.

4 EULER CHARACTERISTIC OF OCEANIC SEA STATES

In the following we extend (11) to deal with the *EC* of excursion sets of spatial snapshots of oceanic sea states measured via VWASS (see fig. 2). To properly represent oceanic nonlinearities (Fedele 2008), we adopt the second order model of Tayfun (1986) and define the wave surface η_{nl} over *S* as

$$\eta_{nl} = \eta + \frac{\mu}{2}\left(\eta^2 - \hat{\eta}^2\right)$$

where $\mu = \lambda_3/3$ is the wave steepness, which relates to the skewness λ_3 of η_{nl}, and $\hat{\eta}$ is the Hilbert transform of a normalized Gaussian field η. The expected *EC* of the excursion set $A_{\eta_{nl},\xi}$ of η_{nl} is equal to the expected *EC* of the excursion set A_{η,ξ_1} of η if the linear threshold ξ_1 relates to the nonlinear ξ through $\xi = \xi_1 + \mu/2\,\xi_1{}^2$. By a variable transformation, from (11) it follows that

$$EC(A_\xi) = A_S\,(2\pi)^{-3/2}|\mathbf{\Lambda}|^{1/2}\,\frac{-1+\sqrt{(-1+2\mu\xi}}{\mu}\exp\left[-\frac{\left(-1+\sqrt{(-1+2\mu\xi}\right)^2}{2\mu^2}\right].$$

Fig. 4 plots the observed *EC* and the expected *EC* (Gaussian & Non-Gaussian) against the threshold *h* for the oceanic video data collected via VWASS. The data agree with the theoretical model.

5 CONCLUSIONS

We have proposed a novel variational image sensor (VWASS) for the stereo reconstruction of wave surfaces. The rich information content of the acquired three-dimensional video data is then exploited to compute reliable estimates the largest crest height over an area using the Euler Characteristic of excursion sets. The wave statistics of large waves over an area is of relevant significance in the offshore industry. In particular, the expectation of high waves over relatively small areas has important implications for a proper design of the air gap under the deck of fixed offshore structures. Localized damages have sometimes been observed on the lower decks of platforms after storms.

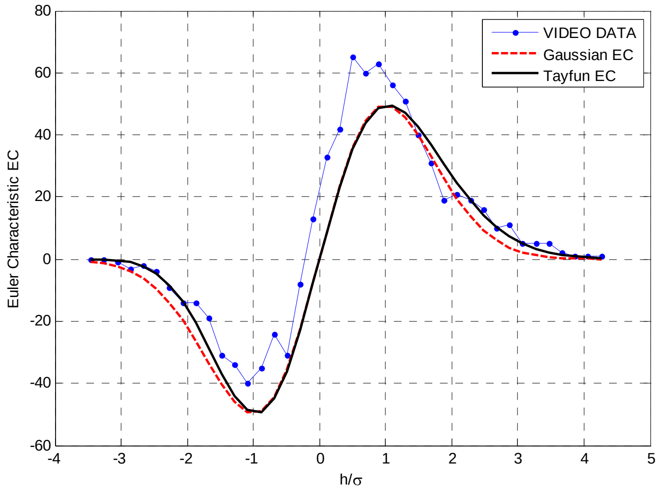

Figura 4. Observed EC and the expected EC against the threshold, as for the oceanic video data collected via VWASS shown in Fig. 2.

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